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NEW IDENTITIES FOR RATIOS OF RAMANUJAN'S THETA FUNCTION

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ABSTRACT. Ramanujan in his notebooks, has established several new modular equation which he denoted as P and Q. In this paper, we establish several new identities for ratios of Ramanujan's theta function involving $\varphi(q)$. We establish some new explicit evaluations for the ratios of Ramanujan's theta function. We also establish some new modular relations for a continued fraction of order twelve $H(q)$ with $H(q^n)$ for n =2, 4, 6, 8, 10, 12, 14 and 16.

1. INTRODUCTION

In Chapter 16, of his second notebook [12], [3, pp. 257-262], Ramanujan develops the theory of theta function and his theta-function is defined by

$$\begin{aligned} f(a,b) &:= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1, \\ &= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \end{aligned}$$

where $(a;q)_\infty := \prod_{n=1}^{\infty}(1-aq^{n-1}), \quad |q|<1$.

Following Ramanujan, we define

$$\begin{aligned} \varphi(q) &:= f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \\ \psi(q) &:= f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \\ f(-q) &:= f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \end{aligned}$$

Now we define a modular equation in brief. The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n, \ \mid x \mid < 1,$$

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where

$$(a)_0 = 1, (a)_n = a(a+1)(a+2)...(a+n-1), \text{for } n \geq 1.$$

Let

$$z := z(x) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$$

and

$$q := q(x) := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right),$$

where $0 < x < 1$.

Let n denote a fixed natural number, and assume that

$$n \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}. \tag{1.1}$$

Then a modular equation of degree n in the classical theory is a relation between α and β induced by (1.1).

In [2], N. D. Baruah has established the relation between

$$P := \frac{\varphi(q)}{\varphi(q^3)} \text{ with } Q := \frac{\varphi(q^n)}{\varphi(q^{3n})},$$

for $n = 5$ and 7. M. S. Mahadeva Naika, S. Chandankumar and K. Sushan Bairy [7], have established the relation connecting P and Q for $n = 9$, 17 and 19. Motivated by these works, we establish several new modular identities connecting P and Q for $n = 2$, 4, 6, 8, 10, 12, 14 and 16.

In Section 2, we collect some results which are useful to prove our main results. In Section 3, we establish several new modular equations of degree 3 for the ratios of Ramanujan's theta function. In Section 4, we establish some explicit evaluations for the ratios of Ramanujan's theta function. In section 5, we establish some new modular relations for a continued fraction of order twelve and its evaluations. In Section 6, we explicitly evaluate Ramanujan's remarkable product of theta functions.

2. Preliminaries

In this section, we collect some identities which are useful in establishing our main results.

Lemma 2.1. [3, Entry 27(i), p. 43] *If $\alpha\beta = \pi$, then*

$$\sqrt{\alpha}\varphi(e^{-\alpha^2}) = \sqrt{\beta}\varphi(e^{-\beta^2}). \tag{2.1}$$

Lemma 2.2. [3, Entry 24(iv), p. 39] *We have*

$$f^3(-q^2) = \varphi(-q)\psi^2(q). \tag{2.2}$$

Lemma 2.3. [3, Entry 10(i), 11(ii), pp. 122–123] *For* $0 < \alpha < 1$,

$$\varphi(q) = \sqrt{z}, \tag{2.3}$$

$$\sqrt{2}q^{1/8}\psi(-q) = \sqrt{z}\{\alpha(1-\alpha)\}^{1/8}. \tag{2.4}$$

Lemma 2.4. [3, Entry 3(xii), pp. 352–353] *Let α, β and γ be of the first, third and ninth degrees respectively. Let m denote the multiplier connecting α, β and m' be the multiplier relating γ, δ, then*

$$\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4} + \left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4} - \left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} = -3\frac{m}{m'}, \tag{2.5}$$

$$\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{1/4} - \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4} = \frac{m'}{m}. \tag{2.6}$$

Lemma 2.5. [3, Entry 11(viii), (ix), p. 384] *Let α, β, γ and δ be of the first, third, fifth and fifteenth degrees respectively. Let m denote the multiplier connecting α and β and m' be the multiplier relating γ and δ. Then*

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} = \sqrt{\frac{m'}{m}}, \tag{2.7}$$

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} = -\sqrt{\frac{m}{m'}}. \tag{2.8}$$

Lemma 2.6. [3, Entry 13(i), (ii), p. 401] *Let α, β, γ and δ be of the first, third, seventh and twenty first degrees respectively. Let m denote the multiplier connecting α and β and m' be the multiplier relating γ and δ. Then*

$$\begin{aligned}\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/4} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/4} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/4}&\\ + 4\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/6} = \frac{m}{m'},&\end{aligned} \tag{2.9}$$

$$\begin{aligned}\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/4} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/4} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/4}&\\ + 4\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/6} = \frac{m'}{m}.&\end{aligned} \tag{2.10}$$

Lemma 2.7. [4, Entry 51, p. 204] *Let* $X = \dfrac{f^2(-q)}{q^{1/6}f^2(-q^3)}$ *and* $Y = \dfrac{f^2(-q^2)}{q^{1/3}f^2(-q^6)}$, *then*

$$XY + \frac{9}{XY} = \left(\frac{Y}{X}\right)^3 + \left(\frac{X}{Y}\right)^3. \tag{2.11}$$

Lemma 2.8. [1, Theorem 5.1] *If* $M = \dfrac{\psi(-q)}{q^{1/4}\psi(-q^3)}$ *and* $N = \dfrac{\varphi(q)}{\varphi(q^3)}$, *then*

$$N^4 + M^4N^4 = 9 + M^4. \tag{2.12}$$

3. New modular equations of degree 3

In this section, we establish several new modular equations of degree 3 for the ratios of Ramanujan's theta function involving $\varphi(q)$.

Theorem 3.1. *If* $P := \dfrac{\varphi(q)\varphi(q^2)}{\varphi(q^3)\varphi(q^6)}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^6)}{\varphi(q^2)\varphi(q^3)}$, *then*

$$P + \frac{3}{P} + Q - \frac{1}{Q} = 4. \tag{3.1}$$

Proof. Equations (2.11) can be rewritten as

$$(UV)^4 + (3UV)^2 = U^6 + V^6, \tag{3.2}$$

where $U := \dfrac{f^2(-q)}{q^{1/6}f^2(-q^3)}$ and $V := \dfrac{f^2(-q^2)}{q^{1/3}f^2(-q^6)}$.
Solving the above equation (3.2) for UV and squaring both sides, we find that

$$2U^6V^6 + 729 + 27U^6 + 27V^6 = M(U^6 + V^6 + 81), \tag{3.3}$$

where $M := \pm\sqrt{81 + 4U^6 + 4V^6}$.

Using equation (2.2) and (2.12), we have

$$U^6 = \frac{X^4(X^4-9)^2}{(1-X^4)^2} \text{ and } V^6 = \frac{Y^4(Y^4-9)^2}{(1-Y^4)^2}, \tag{3.4}$$

where $X := \dfrac{\varphi(q)}{\varphi(q^3)}$ and $Y := \dfrac{\varphi(q^2)}{\varphi(q^6)}$.
Squaring both sides of (3.3) to eliminate M, we find that

$$(X^2Y^2 + X^2 + 3 - 4XY - Y^2)(X^4Y^2 + X^4 - 3Y^2 + Y^4)(X^4Y^4 - X^4 + 9 - Y^4)$$
$$(X^4Y^2 - X^4 - 3Y^2 - Y^4)(X^4 + 3X^2 - X^2Y^4 + Y^4)(X^4 - 3X^2 + X^2Y^4 + Y^4)$$
$$(X^2Y^2 + X^2 + 3 + 4XY - Y^2)(X^2Y^2 - X^2 + 3 + 4XY + Y^2)(X^2Y^2 - X^2 + 3$$

$$- 4XY + Y^2)(X^4Y^4 + 2X^4Y^2 + X^4 - 6X^2 + 9 + 12X^2Y^2 - 6Y^2 + 2X^2Y^4 + Y^4)$$
$$(X^4Y^4 - 2X^4Y^2 + X^4 + 6X^2 + 9 + 12X^2Y^2 + 6Y^2 - 2X^2Y^4 + Y^4) = 0, \tag{3.5}$$

As $q \to 0$, the first factor vanishes faster than the other factors of the equation (3.5). Setting $P := XY$ and $Q := \frac{X}{Y}$ in the first factor, we arrive at the equation (3.1). This completes the proof. □

Theorem 3.2. *If* $P := \frac{\varphi(q)\varphi(q^4)}{\varphi(q^3)\varphi(q^{12})}$ *and* $Q := \frac{\varphi(q)\varphi(q^{12})}{\varphi(q^4)\varphi(q^3)}$, *then*

$$Q^2 + \frac{1}{Q^2} - 16\left[Q + \frac{1}{Q}\right] + P^2 + \frac{3^2}{P^2} + 2\left[Q - \frac{1}{Q}\right]\left[P + \frac{3}{P}\right] + 20 = 0. \tag{3.6}$$

Proof. Equation (3.1) can be written as

$$X^2V^2 + X^2 + 3 - 4XV - V^2 = 0, \tag{3.7}$$

where $X := \frac{\varphi(q)}{\varphi(q^3)}$ and $V := \frac{\varphi(q^2)}{\varphi(q^6)}$.
Note that the above equation (3.7) is unchanged if we change q to q^2, we have

$$V^2Y^2 + V^2 + 3 - 4VY - Y^2 = 0, \tag{3.8}$$

where $Y := \frac{\varphi(q^4)}{\varphi(q^{12})}$. □

Eliminating V from (3.7) and (3.8), we deduce that

$$2Y^2X^4 + Y^4P^4 + X^4 - 16YX^3 + 20Y^2X^2 - 2Y^4X^2 + 6X^2 - 16Y^3X + 9 - 6Y^2 + Y^4 = 0. \tag{3.9}$$

Setting $P := XY$ and $Q := \frac{X}{Y}$ in (3.9), we arrive at the equation (3.6). This completes the proof.

Theorem 3.3. *If* $P := \frac{\varphi(q)\varphi(q^6)}{\varphi(q^3)\varphi(q^{18})}$ *and* $Q := \frac{\varphi(q)\varphi(q^{18})}{\varphi(q^3)\varphi(q^6)}$, *then*

$$9Q^3 - \frac{1}{Q^3} - 36\left[Q^2 + \frac{1}{Q^2}\right] + 9\left[7Q + \frac{1}{Q}\right] - 48 + P^3 + \frac{3^3}{P^3}$$
$$+ \left[P^2 + \frac{3^2}{P^2}\right]\left\{12 - \left[5Q + \frac{3}{Q}\right]\right\} + \left[P + \frac{3}{P}\right]\left\{3\left[Q^2 + \frac{1}{Q^2}\right]\right. \tag{3.10}$$
$$\left. -8\left[3Q + \frac{5}{Q}\right] + 51\right\} = 0.$$

Proof. Employing the equations (2.3), (2.4), (2.5) and (2.6), we deduce that

$$d^2b^2 + a^2d^2 + 3c^2a^2 - b^2c^2 = 0, \tag{3.11}$$

where

$$c = \frac{\varphi(q)}{\varphi(q^3)},\ d = \frac{\varphi(q^3)}{\varphi(q^9)},\ a = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)},\ b = \frac{\psi(-q^3)}{q^{3/4}\psi(-q^9)}.$$

Now, using the equations (2.12), (3.1) and (3.11), we find that

$$\begin{aligned}
&(X^4Y^4 - 2Y^4X^2 + Y^4 + 6Y^2 + 9 + 12Y^2X^2 + 6X^2 - 2Y^2X^4 + X^4)\\
&(27 + 63Y^2X^4 + 153Y^2X^2 + Y^6X^6 + 51X^4Y^4 - 45X^2 + 9X^6 - 27Y^2\\
&- Y^6 + 9X^4 + 9Y^4 + 3Y^2X^6 - 3Y^6X^4 + 3Y^6X^2 + 9Y^4X^2 - 5X^6Y^4\\
&+ 36Y^5X + 120Y^3X - 108YX - 12X^5Y^5 + 40Y^5X^3 + 48Y^3X^3\\
&+ 72X^3Y + 24Y^3X^5 + 36X^5Y)(27 + 63Y^2X^4 + 153Y^2X^2 + Y^6X^6\\
&+ 51X^4Y^4 - 45X^2 + 9X^6 - 27Y^2 - Y^6 + 9X^4 + 9Y^4 + 3Y^2X^6\\
&- 3Y^6X^4 + 3Y^6X^2 + 9Y^4X^2 - 5X^6Y^4 - 36Y^5X - 120Y^3X + 108YX\\
&+ 12X^5Y^5 - 40Y^5X^3 - 48Y^3X^3 - 72X^3Y - 24Y^3X^5 - 36X^5Y) = 0,
\end{aligned} \tag{3.12}$$

where $X := \dfrac{\varphi(q)}{\varphi(q^3)}$ and $Y := \dfrac{\varphi(q^6)}{\varphi(q^{18})}$.

As $q \to 0$, the last factor vanishes of the equation (3.12) vanishes, whereas, the other factors does not vanish. Setting $P := XY$ and $Q := \dfrac{X}{Y}$ in the last factor, we arrive at the equation (3.10).This completes the proof. ☐

Theorem 3.4. *If* $P := \dfrac{\varphi(q)\varphi(q^8)}{\varphi(q^3)\varphi(q^{24})}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^{24})}{\varphi(q^3)\varphi(q^8)}$, *then*

$$\begin{aligned}
&Q^4 + \frac{1}{Q^4} + 80\left[Q^3 - \frac{1}{Q^3}\right] + 320\left[Q^2 + \frac{1}{Q^2}\right] - 80\left[Q - \frac{1}{Q}\right] + 884\\
&+ P^4 + \frac{3^4}{P^4} + 4\left[P^3 + \frac{3^3}{P^3}\right]\left\{4 + \left[Q - \frac{1}{Q}\right]\right\} + 2\left[P^2 + \frac{3^2}{P^2}\right]\\
&\times\left\{16 - 8\left[Q - \frac{1}{Q}\right] + 3\left[Q^2 + \frac{1}{Q^2}\right]\right\} + 4\left[P + \frac{3}{P}\right]\\
&\times\left\{-52 + 22\left[Q - \frac{1}{Q}\right] - 52\left[Q^2 + \frac{1}{Q^2}\right] + \left[Q^3 - \frac{1}{Q^3}\right]\right\} = 0.
\end{aligned} \tag{3.13}$$

Proof. Using the equations (3.1) and (3.6), we obtain (3.13). ☐

Theorem 3.5. *If* $P := \dfrac{\varphi(q)\varphi(q^{10})}{\varphi(q^3)\varphi(q^{30})}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^{30})}{\varphi(q^3)\varphi(q^{10})}$, *then*

$$\begin{aligned}
&Q^6 + \frac{1}{Q^6} + 160\left[Q^5 + \frac{1}{Q^5}\right] + 2468\left[Q^4 + \frac{1}{Q^4}\right] + 160\left[Q^3 + \frac{1}{Q^3}\right] + 14864\\
&+ 1745\left[Q^2 + \frac{1}{Q^2}\right] + 8320\left[Q + \frac{1}{Q}\right] + P^6 + \frac{3^6}{P^6} + \left[P + \frac{3}{P}\right]\Bigg(6\left[Q^5 - \frac{1}{Q^5}\right]\\
&+640\left\{\left[Q^2 - \frac{1}{Q^2}\right] - \left[Q^4 - \frac{1}{Q^4}\right]\right\} + 470\left[Q^3 - \frac{1}{Q^3}\right] - 580\left[Q - \frac{1}{Q}\right]\Bigg)\\
&+ \left[P^2 + \frac{3^2}{P^2}\right]\Bigg(15\left[Q^4 + \frac{1}{Q^4}\right] - 576\left[Q^3 + \frac{1}{Q^3}\right] - 1960\left[Q^2 + \frac{1}{Q^2}\right]\\
&-1600\left[Q + \frac{1}{Q}\right] - 1415\Bigg) + \left[P^3 + \frac{3^3}{P^3}\right]\Bigg(20\left[Q^3 - \frac{1}{Q^3}\right] + 384\left[Q^2 - \frac{1}{Q^2}\right]\\
&+350\left[Q - \frac{1}{Q}\right]\Bigg) + \left[P^4 + \frac{3^4}{P^4}\right]\Bigg(15\left[Q^2 + \frac{1}{Q^2}\right] + 160\left[Q + \frac{1}{Q}\right] + 324\Bigg)\\
&+ 6\left[P^5 + \frac{3^5}{P^5}\right]\left(Q - \frac{1}{Q}\right) = 0.
\end{aligned} \tag{3.14}$$

Proof. Employing the equations (2.3), (2.4), (2.7) and (2.8), we deduce that

$$\frac{a}{b} = \frac{u-v}{u+v}, \tag{3.15}$$

where $u = \dfrac{\varphi(q)}{\varphi(q^3)}$, $v = \dfrac{\varphi(q^5)}{\varphi(q^{15})}$, $a = \dfrac{\psi(-q)}{q^{1/4}\psi(-q^3)}$, $b = \dfrac{\psi(-q^5)}{q^{5/4}\psi(-q^{15})}$.
Using the equations (2.12), (3.1) and (3.15), we find that

$$\begin{aligned}
&(Y^4X^4 - 2Y^2X^4 + X^4 + 6X^2 + 9 + 12Y^2X^2 + 6Y^2 - 2Y^4X^2 + Y^4)\\
&(729 + 1215X^4 + 1458X^2 + 540X^6 + 135X^8 + 18X^{10} + 135Y^8 - 1458Y^2\\
&- 540Y^6 + 15Y^8X^{12} + 350Y^8X^{10} - 1415Y^8X^8 + 580Y^8X^6 + 1745Y^8X^4\\
&- 6Y^{12}X^2 - 1410Y^8X^2 - 1920YX^9 - 5184YX^7 + 160YX^{11} + 10368YX^5\\
&+ 12960YX^3 - 640Y^3X^{11} + 160Y^3X^9 + 1920Y^3X^7 - 14400Y^3X^5\\
&+ 12960Y^3X + 2468Y^2X^{10} + 1410Y^2X^8 - 17640Y^2X^6 + 8320Y^5X^7\\
&+ 640Y^5X^9 + 384Y^7X^{11} - 1600Y^7X^9 + 8320Y^7X^5 - 576Y^5X^{11}\\
&- 14400Y^5X^3 - 1960Y^6X^{10} - 580Y^6X^8 + 14864Y^6X^6 + 1740Y^6X^4\\
&- 17640Y^6X^2 - 1920Y^7X^3 - 5184Y^7X - 10368Y^5X + 9450Y^2X^4\\
&+ 26244Y^2X^2 + 1215Y^4 + 15Y^4X^{12} - 9450Y^4X^2 - 12735Y^4X^4
\end{aligned}$$

$$
\begin{aligned}
&-1740Y^4X^6+470Y^4X^{10}+160Y^9X^{11}-1600Y^9X^7-640Y^9X^5\\
&+1920Y^9X-18Y^{10}+160Y^{11}X^9-384Y^{11}X^7-576Y^{11}X^5+640Y^{11}X^3\\
&+Y^{12}+160Y^{11}X+324Y^{10}X^{10}-350Y^{10}X^8-1960Y^{10}X^6-470Y^{10}X^4 \quad (3.16)\\
&+2468Y^{10}X^2+6Y^{10}X^{12}+Y^{12}X^{12}-6Y^{12}X^{10}+15Y^{12}X^8-20Y^{12}X^6\\
&+15Y^{12}X^4+160Y^9X^3+X^{12}+1745Y^4X^8+6Y^2X^{12}+20Y^6X^{12})=0.
\end{aligned}
$$

where $X := \frac{\varphi(q)}{\varphi(q^3)}$ and $Y := \frac{\varphi(q^{10})}{\varphi(q^{30})}$.
As $q \to 0$, the second factor vanishes of the (3.16), but the first factor does not vanish. Setting $P := XY$ and $Q := \frac{X}{Y}$ in the second factor, we arrive at the equation (3.14). This completes the proof. □

Theorem 3.6. *If* $P := \frac{\varphi(q)\varphi(q^{12})}{\varphi(q^3)\varphi(q^{36})}$ *and* $Q := \frac{\varphi(q)\varphi(q^{36})}{\varphi(q^3)\varphi(q^{12})}$, *then*

$$
\begin{aligned}
&81Q^6+\frac{1}{Q^6}-\frac{288}{Q^5}+252\left[3Q^4+\frac{35}{Q^4}\right]+96\left[9Q^3+\frac{68}{Q^3}\right]+52560\\
&+9\left[1993Q^2-\frac{551}{Q^2}\right]-1152\left[6Q-\frac{49}{Q}\right]+P^6+\frac{3^6}{P^6}-\left[P^5+\frac{3^5}{P^5}\right]\\
&\times\left\{48+2\left[5Q+\frac{3}{Q}\right]\right\}+\left[P^4+\frac{3^4}{P^4}\right]\left\{852+96\left[2Q+\frac{3}{Q}\right]+\left[31Q^2+\frac{15}{Q^2}\right]\right\}\\
&+\left[P^3+\frac{3^3}{P^3}\right]\left\{102\left[13Q-\frac{5}{Q}\right]+96\left[Q^2+\frac{5}{Q^2}\right]-4\left[3Q^3+\frac{5}{Q^3}\right]-2320\right\}\\
&+\left[P^2+\frac{3^2}{P^2}\right]\left\{4761-192\left[32Q+\frac{5}{Q}\right]+8\left[63Q^2-\frac{881}{Q^2}\right]-192\left[3Q^3+\frac{10}{Q^3}\right]\right.\\
&\left.-3\left[27Q^4-\frac{5}{Q^4}\right]\right\}+\left[P+\frac{3}{P}\right]\left\{36\left[231Q-\frac{431}{Q}\right]-96\left[57Q^2-\frac{131}{Q^2}\right]\right.\\
&\left.+30\left[21Q^3+\frac{67}{Q^3}\right]-16\left[27Q^4+\frac{163}{Q^4}\right]+6\left[9Q^5-\frac{1}{Q^5}\right]-21696\right\}=0.
\end{aligned}
\tag{3.17}
$$

Proof. Employing the equations (3.1) and (3.10), we arrive at the equation (3.17). □

Theorem 3.7. *If* $P := \frac{\varphi(q)\varphi(q^{14})}{\varphi(q^3)\varphi(q^{42})}$ *and* $Q := \frac{\varphi(q)\varphi(q^{42})}{\varphi(q^3)\varphi(q^{14})}$, *then*

$$
\begin{aligned}
&Q^8+\frac{1}{Q^8}-504\left[Q^7-\frac{1}{Q^7}\right]+33792\left[Q^6+\frac{1}{Q^6}\right]-100296\left[Q^5-\frac{1}{Q^5}\right]\\
&+649668+118384\left[Q^4+\frac{1}{Q^4}\right]+39816\left[Q^3-\frac{1}{Q^3}\right]+338688\left[Q^2+\frac{1}{Q^2}\right] \quad (3.18)\\
&-664776\left[Q-\frac{1}{Q}\right]+P^8+\frac{3^8}{P^8}-8\left[P^7+\frac{3^7}{P^7}\right]\left\{7-\left[Q-\frac{1}{Q}\right]\right\}
\end{aligned}
$$

$$+\left[P^6+\frac{3^6}{P^6}\right]\left\{832+56\left[Q-\frac{1}{Q}\right]+28\left[Q^2+\frac{1}{Q^2}\right]\right\}+56\left[P^5+\frac{3^5}{P^5}\right]$$
$$\times\left\{69-8\left[Q-\frac{1}{Q}\right]+43\left[Q^2+\frac{1}{Q^2}\right]+\left[Q^3-\frac{1}{Q^3}\right]\right\}+\left[P^4+\frac{3^4}{P^4}\right]$$
$$\times\left\{7728-728\left[Q-\frac{1}{Q}\right]+896\left[Q^2+\frac{1}{Q^2}\right]+2328\left[Q^3-\frac{1}{Q^3}\right]+70\left[Q^4+\frac{1}{Q^4}\right]\right\}$$
$$-8\left[P^3+\frac{3^3}{P^3}\right]\left\{2128\left[Q-\frac{1}{Q}\right]+1897\left[Q^2+\frac{1}{Q^2}\right]-2870\left[Q^3-\frac{1}{Q^3}\right]\right.$$
$$\left.+1333\left[Q^4+\frac{1}{Q^4}\right]-7\left[Q^5-\frac{1}{Q^5}\right]+6335\right\}+28\left[P^2+\frac{3^2}{P^2}\right]$$
$$\times\left\{-3344-2014\left[Q-\frac{1}{Q}\right]-2850\left[Q^2+\frac{1}{Q^2}\right]+2714\left[Q^3-\frac{1}{Q^3}\right]\right.$$
$$\left.-976\left[Q^4+\frac{1}{Q^4}\right]-122\left[Q^5-\frac{1}{Q^5}\right]+\left[Q^6+\frac{1}{Q^6}\right]\right\}+8\left[P+\frac{3}{P}\right]$$
$$\left\{8043-42126\left[Q-\frac{1}{Q}\right]+12999\left[Q^2+\frac{1}{Q^2}\right]+13272\left[Q^3-\frac{1}{Q^3}\right]\right.$$
$$\left.-4263\left[Q^4+\frac{1}{Q^4}\right]-1792\left[Q^5-\frac{1}{Q^5}\right]+847\left[Q^6+\frac{1}{Q^6}\right]+\left[Q^7-\frac{1}{Q^7}\right]\right\}=0. \tag{3.19}$$

Proof. The proof of the identity (3.19) is similar to the proof of the identity (3.14), except that in place of results (2.7) and (2.8), results (2.9) and (2.10) are used. □

Theorem 3.8. *If* $P:=\dfrac{\varphi(q)\varphi(q^{16})}{\varphi(q^3)\varphi(q^{48})}$ *and* $Q:=\dfrac{\varphi(q)\varphi(q^{48})}{\varphi(q^3)\varphi(q^{16})}$, *then*

$$P^8+\frac{3^8}{P^8}+Q^8+\frac{1}{Q^8}-832\left[Q^7+\frac{1}{Q^7}\right]+99296\left[Q^6+\frac{1}{Q^6}\right]-315456\left[Q^5+\frac{1}{Q^5}\right]$$
$$+577008\left[Q^4+\frac{1}{Q^4}\right]-734272\left[Q^3+\frac{1}{Q^3}\right]+1906912\left[Q^2+\frac{1}{Q^2}\right]-2587456$$
$$\times\left[Q+\frac{1}{Q}\right]+4126660+16\left[P+\frac{3}{P}\right]\left\{-28287\left[Q-\frac{1}{Q}\right]-14952\left[Q^2-\frac{1}{Q^2}\right]\right.$$
$$+15456\left[Q^3-\frac{1}{Q^3}\right]+7632\left[Q^4-\frac{1}{Q^4}\right]-4204\left[Q^5-\frac{1}{Q^5}\right]+1160\left[Q^6-\frac{1}{Q^6}\right]$$
$$\left.+\left[\frac{Q^7}{2}-\frac{1}{2Q^7}\right]\right\}+4\left[P^2+\frac{3^2}{P^2}\right]\left\{-28112\left[Q+\frac{1}{Q}\right]-33454\left[Q^2+\frac{1}{Q^2}\right]+42056\right.$$
$$\left.+2128\left[Q^3+\frac{1}{Q^3}\right]-9640\left[Q^4+\frac{1}{Q^4}\right]-11312\left[Q^5+\frac{1}{Q^5}\right]+7\left[Q^6+\frac{1}{Q^6}\right]\right\}$$
$$+8\left[P^3+\frac{3^3}{P^3}\right]\left\{-5768\left[Q-\frac{1}{Q}\right]-2512\left[Q^2-\frac{1}{Q^2}\right]+6758\left[Q^3-\frac{1}{Q^3}\right]\right.$$

$$+1760\left[Q^4-\frac{1}{Q^4}\right]+7\left[Q^5-\frac{1}{Q^5}\right]\Bigg\}+2\left[P^4+\frac{3^4}{P^4}\right]\Bigg\{3360\left[Q+\frac{1}{Q}\right]+9432$$
$$-7472\left[Q^2+\frac{1}{Q^2}\right]+6048\left[Q^3+\frac{1}{Q^3}\right]+35\left[Q^4+\frac{1}{Q^4}\right]\Bigg\}+8\left[P^5+\frac{3^5}{P^5}\right]$$
$$\times\Bigg\{-528\left[Q-\frac{1}{Q}\right]-240\left[Q^2-\frac{1}{Q^2}\right]+7\left[Q^3-\frac{1}{Q^3}\right]\Bigg\}+4\left[P^6+\frac{3^6}{P^6}\right]$$
$$\times\Bigg\{-208\left[Q+\frac{1}{Q}\right]+7\left[Q^2+\frac{1}{Q^2}\right]+584\Bigg\}+8\left[P^7+\frac{3^7}{P^7}\right]\left[Q-\frac{1}{Q}\right]=0. \tag{3.20}$$

Proof. Using the equations (3.1) and (3.13), we obtain the equation (3.20). □

4. Evaluations for the ratios of Ramanujan's theta function

In this section, by using the modular equations obtained in the previous section 3, we establish some explicit evaluations for the ratios of Ramanujan's theta function.

Theorem 4.1. *We have*

$$\frac{\varphi(e^{-2\pi/\sqrt{3}})}{\varphi(e^{-2\pi\sqrt{3}})}=\frac{3^{1/4}(\sqrt{3}+1)(\sqrt{2}-1)}{\sqrt{2}}, \tag{4.1}$$

$$\frac{\varphi(e^{-\pi/2\sqrt{3}})}{\varphi(e^{-\pi\sqrt{3}/2})}=\frac{3^{1/4}(\sqrt{3}-1)(\sqrt{2}+1)}{\sqrt{2}}, \tag{4.2}$$

$$\frac{\varphi^2(e^{-\pi\sqrt{2/3}})}{\varphi^2(e^{-\pi\sqrt{6}})}=3\sqrt{2}-2\sqrt{3}+3-\sqrt{6}, \tag{4.3}$$

$$\frac{\varphi^2(e^{-\pi/\sqrt{6}})}{\varphi^2(e^{-\pi\sqrt{3/2}})}=3\sqrt{2}+2\sqrt{3}-3-\sqrt{6}, \tag{4.4}$$

$$\frac{\varphi^2(e^{-\pi\sqrt{2}})}{\varphi^2(e^{-3\pi\sqrt{2}})}=21+9\sqrt{6}-15\sqrt{2}-12\sqrt{3}, \tag{4.5}$$

$$\frac{\varphi^2(e^{-\pi/3\sqrt{2}})}{\varphi^2(e^{-\pi/\sqrt{2}})}=\sqrt{6}+\sqrt{2}-1, \tag{4.6}$$

$$\frac{\varphi^2(e^{-2\pi\sqrt{2/3}})}{\varphi^2(e^{-2\pi\sqrt{6}})}=\sqrt{3}\left[\sqrt{596+344\sqrt{3}-8A}-\sqrt{595+344\sqrt{3}-8A}\right], \tag{4.7}$$

$$\frac{\varphi^2(e^{-\pi/2\sqrt{6}})}{\varphi^2(e^{-\sqrt{3}\pi/2\sqrt{2}})}=\sqrt{3}\left[\sqrt{596+344\sqrt{3}-8A}+\sqrt{595+344\sqrt{3}-8A}\right], \tag{4.8}$$

where $A:=(12+7\sqrt{3})(\sqrt{19+11\sqrt{3}})$.

$$\frac{\varphi(e^{-4\pi/\sqrt{3}})}{\varphi(e^{-4\pi\sqrt{3}})}=3^{1/4}\left[\sqrt{\frac{103+42\sqrt{6}-2B}{2}}-\sqrt{\frac{101+42\sqrt{6}-2B}{2}}\right], \tag{4.9}$$

$$\frac{\varphi(e^{-\pi/4\sqrt{3}})}{\varphi(e^{-\pi\sqrt{3}/4})} = 3^{1/4}\left[\sqrt{\frac{103+42\sqrt{6}-2B}{2}}+\sqrt{\frac{101+42\sqrt{6}-2B}{2}}\right], \tag{4.10}$$

where $B := \sqrt{5188+2118\sqrt{6}}$.

$$\begin{aligned}\frac{\varphi^2(e^{-\pi\sqrt{14/3}})}{\varphi^2(e^{-\pi\sqrt{42}})} =&\sqrt{3}\left[\sqrt{16352-9440\sqrt{3}+3568\sqrt{21}-6180\sqrt{7}}\right.\\ &\left.-\sqrt{16351-9440\sqrt{3}+3568\sqrt{21}-6180\sqrt{7}}\right],\end{aligned} \tag{4.11}$$

$$\begin{aligned}\frac{\varphi^2(e^{-\pi/\sqrt{42}})}{\varphi^2(e^{-\pi\sqrt{3/14}})} =&\sqrt{3}\left[\sqrt{16352-9440\sqrt{3}+3568\sqrt{21}-6180\sqrt{7}}\right.\\ &\left.+\sqrt{16351-9440\sqrt{3}+3568\sqrt{21}-6180\sqrt{7}}\right],\end{aligned} \tag{4.12}$$

$$\frac{\varphi^2(e^{-\pi/6})}{\varphi^2(e^{-\pi/2})} = (\sqrt{6}-\sqrt{3})(3^{1/4}+1)+3^{1/4}, \tag{4.13}$$

$$\frac{\varphi^2(e^{-2\pi})}{\sqrt{3}\varphi^2(e^{-6\pi})} = 39+22\sqrt{3}-\sqrt{2a}+\sqrt{2\left[3009+1737\sqrt{3}-\frac{115713\sqrt{2}+66807\sqrt{6}}{\sqrt{a}}\right]}, \tag{4.14}$$

where $a := 1479+854\sqrt{3}$.

$$\frac{\varphi^2(e^{-\pi\sqrt{10/3}})}{\sqrt{3}\varphi^2(e^{-\pi\sqrt{30}})} = 2\sqrt{10b}+(13\sqrt{3}-22)-\sqrt{1968-1136\sqrt{3}-\sqrt{10}\frac{(4486-2590\sqrt{3})}{\sqrt{b}}}, \tag{4.15}$$

$$\frac{\varphi^2(e^{-\pi/\sqrt{30}})}{\sqrt{3}\varphi^2(e^{-\pi\sqrt{3/10}})} = 2\sqrt{10b}+(13\sqrt{3}-22)+\sqrt{1968-1136\sqrt{3}-\sqrt{10}\frac{(4486-2590\sqrt{3})}{\sqrt{b}}}, \tag{4.16}$$

where $b := 26-15\sqrt{3}$.

Proof of (4.1). Putting $\alpha = \sqrt{\pi/\sqrt{3}}$ and $\beta = \sqrt{\pi\sqrt{3}}$ in (2.1), we find that

$$\frac{\varphi(e^{-\pi/\sqrt{3}})}{\varphi(e^{-\pi\sqrt{3}})} = 3^{1/4}. \tag{4.17}$$

Using the equation (4.17) in the equation (3.1), with $q = e^{-\pi/\sqrt{3}}$, we deduce that

$$3^{1/4}T+\frac{3}{3^{1/4}T}+\frac{3^{1/4}}{T}-\frac{T}{3^{1/4}} = 4, \quad \text{where } T = \frac{\varphi(e^{-2\pi/\sqrt{3}})}{\varphi(e^{-2\pi\sqrt{3}})}. \tag{4.18}$$

Solving the above equation (4.18) for T and $1 < T < 2$, we arrive at the equation (4.1). □

Proof of (4.2). Setting $q = e^{-\pi/2\sqrt{3}}$ in the equation (3.1), then using the equation (4.17), we deduce that

$$t^2 - 2t3^{3/4} + 2t3^{1/4} + 2\sqrt{3} - 3 = 0, \quad \text{where } t = \frac{\varphi(e^{-\pi/2\sqrt{3}})}{\varphi(e^{-\pi\sqrt{3}/2})}. \tag{4.19}$$

Solving the above equation (4.19) for t and $1 < t < 2$, we arrive at the equation (4.2). □

Since the proofs of the identities (4.3)–(4.16) are similar to the proofs of the identities (4.1) and (4.2). Hence we omit the details.

5. Modular Identities for a continued fraction of order twelve

In [11], M. S. Mahadeva Naika, B. N. Dharmendra and K. Shivashankara have established a continued fraction of order twelve. They have also established several explicit evaluations, reciprocity theorems and integral representations.

In this section, we establish some new modular relations for a continued fraction of order twelve using the identities established in Section 3. We also establish some explicit evaluations for a continued fraction of order twelve.

Lemma 5.1. [6], [11, Theorem 3.1]*We have*

$$\frac{\varphi(q)}{\varphi(q^3)} = \frac{1+H(q)}{1-H(q)}, \quad \textit{where} \quad H(q) = \frac{qf(-q,-q^{11})}{f(-q^5,-q^7)}. \tag{5.1}$$

Theorem 5.1. *If* $u := H(q)$ *and* $v := H(q^2)$*, then*

$$(u+v)^2 = u^2v + v. \tag{5.2}$$

Proof. Using the equations (3.1) and (5.1), we obtain the equation (5.2). □

Theorem 5.2. *If* $u := H(q)$ *and* $v := H(q^4)$*, then*

$$(1-v)(1+v^2)(v-u^4) + (2v^3 + 2v - 8v^2)u^2 + 4uv(v-1)(1-vu^2) = 0. \tag{5.3}$$

Proof. Using the equations (3.6) and (5.1), we obtain the equation (5.3). □

Theorem 5.3. *If* $u := H(q)$ *and* $v := H(q^6)$*, then*

$$\begin{aligned}&(v^5 + v^2 - 1 - v^3)(u^6 - v) + 6(-v^4 + v^5 - v + v^2)(u - u^5)\\&+ 9(1 - 3v - v^3 + 3v^2)(u^2v - v^2u^4) + 2(8v^2 + 8v^4 - 18v^3 - v^5 - v)u^3 = 0.\end{aligned} \tag{5.4}$$

Proof. Using the equations (3.10) and (5.1), we obtain the equation (5.4). □

Theorem 5.4. *If $u := H(q)$ and $v := H(q^8)$, then*

$$\begin{aligned}&(v^7 - 3v^2 + v - v^6 + 3v^5 - 3v^4 - 1 + 3v^3)(u^8 - v) + 8(v - 3v^5 - v^3 - v^7\\&+ 2v^4 + 2v^6)u^7 + 4(14v^5 + 5v^7 + 3v^2 + 5v^3 - 10v^4 - 13v^6 - 4v)u^6\\&+ 8(v^2 + v + 12v^4 - 10v^3 + 3v^6 - 5v^5 - 2v^7)u^5 + 2(v + v^7 + 35v^3 + 35v^5\\&- 72v^4 - 4v^6 - 4v^2)u^4 + 8(v^7 - 2v + 12v^4 - 5v^3 + 3v^2 - 10v^5 + v^6)u^3\\&+ 4(3v^6 - 13v^2 + 14v^3 - 10v^4 + 5v + 5v^5 - 4v^7)u^2 + 8(v^7 + 2v^2 + 2v^4\\&- v - v^5 - 3v^3)u = 0.\end{aligned} \tag{5.5}$$

Proof. Using the equations (3.13) and (5.1), we obtain the equation (5.5). □

Theorem 5.5. *If $u := H(q)$ and $v := H(q^{10})$, then*

$$\begin{aligned}&[1 + 15v^2 - 6v^5 + v^6 - 20v^3 + 15v^4 - 6v][u^{12}v^5 + v] + 2v^5[-1 - 55v^4 + 30v^5\\&- 35v^2 + 6v + 60v^3 - 5v^6]u^{11} + [-185v^8 - 1 - 50v^2 + 35v^{11} - 190v^4 + 11v\\&- 190v^{10} - 56v^6 + 125v^3 + 80v^7 + 176v^5 + 245v^9]u^{10} + 2v^2[-140v + 20 + 340v^2\\&+ 40v^6 - 95v^7 + 110v^8 - 115v^5 - 425v^3 + 290v^4 - 25v^9]u^9 + [-1020v^4 - 160v^8\\&+ 60v^2 - 20v - 1630v^6 + 865v^7 + 1675v^5 + 230v^3 - 130v^{10} + 25v^{11} + 105v^9]u^8\\&+ 4v[10 + 5v^8 + 100v^7 + 280v^3 + 678v^5 - 415v^6 - 575v^4 + 2v^{10} - 10v^9 - 40v\\&- 35v^2]u^7 + 2v[-10v^8 + 95v^9 - 10v^2 - 420v^7 + 95v + 1150v^4 - 20v^{10} + 1150v^6\\&- 1622v^5 - 420v^3 - 20]u^6 + 4v[2 - 10v + 280v^7 - 575v^6 + 100v^3 + 10v^{10} + 678v^5\\&- 35v^8 - 415v^4 - 40v^9 + 5v^2]u^5 + [230v^9 - 20v^{11} - 160v^4 - 130v^2 + 25v + 60v^{10}\\&+ 105v^3 + 1675v^7 + 865v^5 - 1020v^8 - 1630v^6]u^4 + 2v[110v - 425v^6 - 25 - 95v^2\\&- 115v^4 - 140v^8 + 40v^3 + 340v^7 + 20v^9 + 290v^5]u^3 + [245v^3 - 190v^2 + 35v\\&- 190v^8 - 56v^6 + 11v^{11} + 176v^7 + 80v^5 - 185v^4 + 125v^9 - 50v^{10} - v^{12}]u^2\\&+ [-2v^7 + 60v^2 + 120v^4 - 110v^3 - 70v^5 + 12v^6 - 10v]u = 0.\end{aligned} \tag{5.6}$$

Proof. Using the equations (3.14) and (5.1), we obtain the equation (5.6). □

Remark 1. Similarly, one can establish modular relation for $H(q)$ with $H(q^{12})$ using the equation (3.17), $H(q)$ with $H(q^{14})$ using the equation (3.19) and $H(q)$ with $H(q^{16})$ using the equation (3.20).

In the following theorem, we establish some new explicit evaluations for a continued fraction of order twelve $H(q)$, by using the evaluations of ratios of theta function established in section 4.

Theorem 5.6. *We have*

$$H(e^{-2\pi/\sqrt{3}}) = \frac{3^{1/4}(\sqrt{3}+1)(\sqrt{2}-1)-\sqrt{2}}{3^{1/4}(\sqrt{3}+1)(\sqrt{2}-1)+\sqrt{2}}, \tag{5.7}$$

$$H(e^{-\pi/2\sqrt{3}}) = \frac{3^{1/4}(\sqrt{3}-1)(\sqrt{2}+1)-\sqrt{2}}{3^{1/4}(\sqrt{3}-1)(\sqrt{2}+1)+\sqrt{2}}, \tag{5.8}$$

$$H(e^{-\pi\sqrt{2/3}}) = \frac{\sqrt{(\sqrt{6}+\sqrt{3})(\sqrt{3}-\sqrt{2})}-1}{\sqrt{(\sqrt{6}+\sqrt{3})(\sqrt{3}-\sqrt{2})}+1}, \tag{5.9}$$

$$H(e^{-\pi/\sqrt{6}}) = \frac{\sqrt{(\sqrt{6}-\sqrt{3})(\sqrt{3}+\sqrt{2})}-1}{\sqrt{(\sqrt{6}-\sqrt{3})(\sqrt{3}+\sqrt{2})}+1}, \tag{5.10}$$

$$H(e^{-\pi\sqrt{2}}) = \frac{\sqrt{21+9\sqrt{6}-15\sqrt{2}-12\sqrt{3}}-1}{\sqrt{21+9\sqrt{6}-15\sqrt{2}-12\sqrt{3}}+1}, \tag{5.11}$$

$$H(e^{-\pi/3\sqrt{2}}) = \frac{\sqrt{\sqrt{6}+\sqrt{2}-1}-1}{\sqrt{\sqrt{6}+\sqrt{2}-1}+1}, \tag{5.12}$$

$$H(e^{-\pi/6}) = \frac{\sqrt{\sqrt[4]{3}+(\sqrt[4]{3}+1)(\sqrt{6}-\sqrt{3})}-1}{\sqrt{\sqrt[4]{3}+(\sqrt[4]{3}+1)(\sqrt{6}-\sqrt{3})}+1}. \tag{5.13}$$

Proof. Rewriting equation (5.1) and set $q = e^{-\pi\sqrt{4/3}}$ and using (4.1), we arrive at equation (5.7).

Since the proofs of the identities (5.8)–(5.13) are similar to the proof of the identity (5.7). Hence we omit the details. □

Remark 2. One can explicitly evaluate $H(e^{-\pi\sqrt{n/3}})$ for n =8, 1/8, 10, 1/10, 12, 14, 1/14, 16 and 1/16.

6. Ramanujan's remarkable product of theta functions

On page 338 in his first notebook, Ramanujan defines

$$a_{m,n} := \frac{ne^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}}\psi^2(e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})},$$

and offers a list of eighteen particular values. All these eighteen values have been established by B. C. Berndt, H. H. Chan and L. C. Zhang [5]. In [10], Mahadeva Naika and B. N. Dharmendra have established some new general theorems for the explicit evaluations of Ramanujan's remarkable product of theta-functions $a_{m,n}$. For more details one can see [8] and [9].

In this section, we evaluate some new explicit evaluations of Ramanujan's remarkable product of theta functions $a_{3,m}$ for m =2, 4, 6, 8, 10, 12, 14 and 16.

Lemma 6.1. [10, Theorem 2.4] *If* $Q = \dfrac{\varphi(e^{-\pi\sqrt{\frac{m}{3}}})}{\varphi(e^{-\pi\sqrt{3m}})}$, *then*

$$a_{3,m}^2 = \frac{9(1-Q^4)}{Q^4(Q^4-9)}, \quad Q^4 \neq 9, \tag{6.1}$$

where m is any positive rational.

Theorem 6.1. *We have*

$$a_{3,2}^2 = (\sqrt{3}+\sqrt{2})(3-2\sqrt{2}), \tag{6.2}$$

$$a_{3,4}^2 = \frac{(11\sqrt{3}-19)(3+2\sqrt{2})}{\sqrt{2}}, \tag{6.3}$$

$$a_{3,6}^2 = \frac{4\sqrt{2}+\sqrt{6}-2-\sqrt{3}}{9}. \tag{6.4}$$

Proof of (6.2)–(6.4). Using equation (6.1) along with (4.3), (4.1) and (4.5), we arrive at (6.2), (6.3) and (6.4) respectively. This completes the proof. □

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